σ-Extension of locally solid topologies

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Summary.
1. Introduction.
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Summary.

Let L be an almost σ-Dedekind complete Riesz space. In [9] I. Labuda has shown that if (L, τ) has the σ-Fatou property, then there exists the largest σ-enlargement $(\tilde{L}, \tau^{\sim})$ of (L, τ) that is a unique σ-Nakano space. In this paper we give a sequential version of Theorem 3.7 given in [2]([1, 24.3]), and the results tell us what Hausdorff locally solid σ-Lebesgue topologies on L extend to a σ-Lebesgue topology on L^s.

1. Introduction.

For notation and basic terminology concerning Riesz spaces not explained below, we refer to the books [1], [11] and [14]. Let L be a Riesz space and $L^+ = (x \in L : x \geq \theta)$. L is called *σ-laterally complete* if every positive disjoint sequence of L has the supremum in L. L is called *σ-universally complete* if it is σ-Dedekind complete and σ-laterally complete. A σ-universal completion of a Riesz space L is a σ-universally complete Riesz space K having a super order dense Riesz subspace M that is Riesz isomorphic to L. As usual, identifying L with its image M in K, we shall treat L as a Riesz subspace of K. L is called *almost σ-Dedekind complete* if it is Riesz isomomrphic to a super order dense Riesz subspace of some σ-Dedekind complete Riesz space. By [1, 23.27] L is almost σ-Dedekind complete if and only if L has a σ-universal completion which is determined uniquely up to a Riesz isomorphism. The σ-universal completion of L denotes by L^s. If L is almost σ-Dedekind complete, then the ideal of L^s generated by L is precisely the σ-Dedekind completion L^σ of L.

Let L be a Riesz space and $x \in L$. Denote by $\boldsymbol{B}_x$ the principal band of L generated by x. If L has the principal projection property, then $L = B_x \oplus B_x^d$ holds for all $x \in L$.

The projection determined by a principal projection band B_x will be denoted by P_x, i.e., $P_x(y) = \sup_n (y \wedge n|x|)$ for all $y \in L^+$. Every σ-Dedekind complete Riesz space has the principal projection property and every σ-laterally complete Riesz space is so.

Let (L, τ) be a locally solid Riesz space. Then by [1, 6.3] there exists a family (ρ_α) of Riesz pseudonorms on L that generates the topology τ. We denote by P_τ the family (ρ_α). A locally solid topology τ on L is called *σ-Lebesque* if $x_n \xrightarrow{(0)} \theta$ in L implies $x_n \xrightarrow{\tau} \theta$, and *$\sigma$-Fatou* if τ has a basis for θ consisting of solid and σ-order closed sets. Using [7, 22 C] it is easy to see that τ is a σ-Lebesque topology if and only if $x_n \xrightarrow{(0)} \theta$ in L implies $\rho(x_n) \rightarrow 0$ for any $\rho \in P_\tau$. Also, by [1, 11.4] it follows that τ is a σ-Fatou topology if and only if $\theta \leq x_n \uparrow x$ in L implies $\rho(x_n) \uparrow \rho(x)$ for all $\rho \in P_\tau$. A locally solid topology τ on L is called *σ-Levi* if for every increasing τ-bounded sequence $(x_n) \subset L^+$, $\sup_n x_n$ exists in L. (L, τ) is called *σ-Nakano space* if τ has both the σ-Levi property and the σ-Fatou property.

Let (L, τ) be a locally convex-solid Riesz space. Then by [1, 6.1] there exists a family (ρ_α) of Riesz seminorms on L that generates the topology τ. We denote by S_τ the family (ρ_α). By replacing the word "locally solid" ("Riesz pseudonorm") with "locally convex-solid" ("Riesz seminorm") in the above, we obtain similarly a locally convex-solid σ-Lebesque topology, σ-Fatou topology, etc.

In [9] I. Labuda has shown that if L is an almost σ-Dedekind complete Riesz space and if τ is a Hausdorff σ-Fatou topology on L, then there exists the largest σ-enlargement $(\tilde{L}, \tau^\sim)$ of (L, τ), in which (L, τ) is embedded super order densely and which is a unique σ-Nakano space, where $L \subset L^\sigma \subset \tilde{L} \subset L^s$. From this it is natural to ask when a Hausdorff σ-Fatou topology on L can be extended to a Hausdorff σ-Fatou topology on L^s. Clearly such a topology must be necessarily σ-Lebesque by [1, 24.1]. In this paper we will show what Hausdorff locally solid σ-Lebesque topologies on L have such an extension.

2. Extension of Riesz homomorphisms.

Let L be an almost σ-Dedekind complete Riesz space and let $L^\sim$ be its order dual. $L_c^\sim$ denots the band in $L^\sim$ consisting of all σ-order continuous linear functionals on L. For each $u \in L^s$ put $L_u = L_{|u|} = (x \in L : |x| \leq |u|)$ and for each $f \in L_c^\sim$ define

$$L_f = \{ u \in L^s : \sup (|f(x)| : x \in L_u) < \infty \}.$$

Since L is super order dense in L^s, L_f is expressible as the set consisting of all elements u of L^s such that $\sup_n |f|(x_n) < \infty$ for a sequence (x_n) of L^+ with $\theta \leq x_n \uparrow |u|$ in L^s. It is easy to show that L_f is a solid Riesz subspace of L^s and $L^\sigma \subset L_f \subset L^s$.

Let $\theta \leq f \in L_c^\sim$. For each $\theta \leq u \in L_f$ put $f^\sim(u) = \sup_n f(x_n)$, where $x_n \in L$ and $\theta \leq x_n \uparrow u$ in L^s, and define $f^\sim(u) = f^\sim(u^+) - f^\sim(u^-)$ for all $u \in L_f$. Since $u^+, u^- \in L_f$, $f^\sim$ is well-defined and it follws by [1, 3.1] that $f^\sim$ is a unique positive linear extension of f on L_f. To see that $f^\sim$ is σ-order continuous on L_f, let $\theta \leq u_n \uparrow u$ in L_f. For each n choose a sequence (x_k^n) of L_{u_n} such that $\theta \leq x_k^n \uparrow u_n$ in L_f, and put $y_n = \vee_{l=1}^n x_n^l$. Then it follows immediately that $\theta \leq y_n \leq u_n$ for all n, $(y_n) \subset L_u$ and $\theta \leq y_n \uparrow u$ in L_f. Hence

$$f^\sim(u) = \sup_n f^\sim(y_n) = \sup_n f^\sim(u_n)$$

holds, and so $f^\sim$ is σ-order continuous on L_f.

In general, if $f \in L_c^\sim$, let $f = f^+ - f^-$ and define $f^\sim$ by $f^\sim(u) = (f^+)^\sim(u) - (f^-)^\sim(u)$ for all $u \in L_f$. Since $\theta \leq f^+, f^- \in L_c^\sim$ and $L_f = L_{|f|} = L_{f^+} \cap L_{f^-}$, it is simple to verify that $f^\sim$ is a unique σ-order continuous linear extension of f on L_f. Thus $f^\sim \in (L_f)_c^\sim$ holds for all $f \in L_c^\sim$. We now have the following result.

THEOREM 1. *Let L be an almost σ-Dedekind complete Riesz space and let π be a Riesz σ-homomorphism on L. Then π can extend uniquely to a Riesz σ-homomorphism $\pi^\sim$ on L^s.*

Proof. From the argument above π has a unique σ-order continuous linear extension $\pi^\sim$ to all of L_π. Further, to show that $\pi^\sim$ is a Riesz homomorphism on L_π let $u \wedge v = \theta$ in L_π. Choose two sequences (x_n) and (y_n) of L such that $\theta \leq x_n \uparrow u$ and $\theta \leq y_n \uparrow v$ in L_π. Since $x_n \wedge y_n = \theta$ for all n, it is immediate that

$$\pi^\sim(u) \wedge \pi^\sim(v) = \sup_n \pi(x_n) \wedge \pi(y_n) = \sup_n \pi(x_n \wedge y_n) = 0.$$

Hence $\pi^\sim$ is a Riesz σ-homomorphism on L_π. Now, we have to show that $L_\pi = L^s$. To this end, assume first that L is σ-Dedekind complete and $\pi^{\sim}(u)=\infty$ for some $\theta<u\in L^s$. Then there exists an element $\theta<x\in L_u$ such that $\theta<\pi(x)<\infty$. Let B_x be the band of L^s generated by x, and put $u=u_1+u_2\in L^s=B_x\oplus B_x^d$. It is not difficult to verify that

$$\theta<\pi(x)<\pi^{\sim}(u_1)=\infty \text{ and } \pi^{\sim}(u_2)=0.$$

Put $e_n=(nx-u_1)^+$ for each $n\in\mathbf{N}$. Since L is Archimedean, there exists an index n_0 such that $e_{n_0}>\theta$, and $\theta\leq e_n\uparrow$ holds in L. By passing to a subsequence (if necessary) we may assume that $\theta<e_n\in L$ for all n. For each n put $v_n=P_{e_n}(u_1)$. Since $(u_1\wedge ke_n)\wedge(nx-u_1)^-=\theta$ for all $k\in\mathbf{N}$, it is immediate that $v_n\wedge(u_1-nx)\leq\theta$ and so $v_n\leq(v_n+nx)\wedge u_1\leq nx$. Hence $v_n\in L$ holds for all n, and $u_1=\sup_n v_n$. Indeed, let $v_0=\sup_n v_n$ and $w_0=u_1-v_0$. Since $\theta\leq w_0\leq u_1-v_n$, $w_0\wedge v_n=\theta$ holds and it follows that $w_0\wedge e_n=\theta$ for all n. From this it is simple to verify that $(w_0+u_1)\wedge nx\leq u_1$ holds for all $n\in\mathbf{N}$. Hence

$$(w_0+u_1)=\sup_n(w_0+u_1)\wedge nx\leq u_1$$

and so $w_0=\theta$. This means that $u_1=\sup_n v_n$. Since $\pi^{\sim}(u_1)=\sup_n\pi(v_n)=\infty$, there exists an index n_0 such that $0<\pi(v_{n_0})<\infty$. Let $e=v_{n_0}$. For each $x\in L^+$ put $\varphi(x)=\sup_k\pi(x\wedge ke)$ and define $\varphi(x)=\varphi(x^+)-\varphi(x^-)$ for all $x\in L$. Then φ is well-defined and positive linear. Since $\theta\leq\varphi\leq\pi$ holds on L, φ is a Riesz σ-homomorphism on L, and it then follows from the argument given in [12, 4.4] that $\varphi=\lambda\pi$ for some $\lambda>0$. Choose a sequence (x_n) of L^+ such that $\theta\leq x_n\uparrow u=u_1+u_2$. Since $u\wedge ke=e$ for each $k\in\mathbf{N}$, it is immediate that $\sup_n\pi(x_n\wedge ke)=\pi(u\wedge ke)=\pi(e)<\infty$, and we get

$$\varphi^{\sim}(u)=\sup_n\varphi(x_n)\leq\pi(e)<\infty,$$

contradicting the fact that $\varphi^{\sim}(u)=\lambda\pi^{\sim}(u)=\infty$. This show that $L_\pi=L^s$ holds. Finally, if L is almost σ-Dedekind complete, let π' be the restriction of $\pi^{\sim}$ to L^σ. π' is a Riesz σ-homomorphism on L^σ and it then follows by the above discussion that π' extends to all of L^s. Hence π extends uniquely to all of L^s as a Riesz σ-homomorphism, and the proof is finished.

Remark. Using the argument similar to preceding proof we obtain the net analog of the previous theorem :

Let L be a Dedekind complete Riesz space and let L^u be its universally completion. If π is an order continuous Riesz homomorphism on L, then there exists a unique order continuous Riesz homomorphism on L^u which is an extension of π.

3. Extension of (L,τ).

Let L be an almost σ-Dedekind complete Riesz space. If τ is a locally (convex) solid σ-Lebesgue topology on L, then by [1, 11.11] τ can uniquely extend to a locally (convex) solid σ-Lebesgue topology τ^σ on L^σ. The following theorem tells us what Hausdorff locally convex-solid σ-Lebesgue topologies on L can be extended to a locally convex-solid topology on L^s.

THEOREM 2. *Let L be an almost σ-Dedekind complete Riesz space and let τ be a Hausdorff locally convex-solid σ-Lebesgue topology on L. Then the following statements are equivalent :*

(1) *τ extends to a locally convex-solid topology on L^s.*

(2) *τ extends to a locally convex-solid σ-Lebesgue topology on L^s.*

(3) *For every $\rho \in S_\tau$ there exist some Riesz σ-homomorphisms π_k, $k = 1,2,...,n$ and some positive real numbers a_k, $k = 1,2,...,n$ (depending upon ρ) such that $a_k \pi_k \le \rho$ on L^+ holds for each k and $\rho \le \Sigma_{k=1}^n a_k \pi_k$ on L^+.*

(4) *For any Hausdorff locally solid σ-Lebesgue topology λ defined on L, all sequences $(x_n : n \in \mathbf{N}) \subset L$ with $x_n \xrightarrow{\lambda} \theta$ in L implies that $x_n \xrightarrow{\tau} \theta$ in L.*

(5) *For any Hausdorff locally solid σ-Lebesgue topology λ defined on L, every λ-bounded subset of L implies τ-bounded.*

(6) *Every dominable subset of L is τ-bounded.*

Proof. (1)$\Rightarrow$(2) Let τ^s be the locally convex-solid topology on L^s that is an extension of τ. By [1, 23.6] if $u_n \downarrow \theta$ in L^s, then for every $\varepsilon > 0$ there exists $\theta \le u \in L^s$ (depending upon ε) such that $\theta \le u_n \le 2^{-n} u + \varepsilon u_1$ for all n. Hence $\rho(u_n) \le 2^{-n} \rho(u) + \varepsilon \rho(u_1)$ holds for each $\rho \in S_{\tau^s}$. This means that (L^s, τ^s) has the σ-Lebesgue property.

(2)$\Rightarrow$(3) Let τ^s be a locally convex-solid σ-Lebesgue topology on L^s that induces τ on L. Let $\rho \in S_{\tau^s}$ and put $A = (u \in L^s : \rho(u) > 0)$. Then it follows that every disjoint subset of positive elements of A is finite. Indeed, if (u_n) is any disjoint sequence of positive elements of A, then $u = \sup_n nu_n/\rho(u_n)$ exists in L^s and $\rho(nu_n/\rho(u_n)) = n \le \rho(u)$ holds for all n. This is a contradiction. Hence there is no proper infinite disjoint sequence of positive elements of A. Let e_i, $i = 1,2,...,n$ be a maximal disjoint system of positive elements in A. Put $e = \sup_{1 \le i \le n} e_i$ and let $L^s = B_e \oplus B_e^d$ for the band B_e of L^s generated by e. It is clear that $\rho(u) = 0$ for every $u \in B_e^d$, and $\rho(u) = \rho(P_e(u))$ holds for all $u \in L^s$. Since B_e is a σ-universally complete Riesz space in its own rigth, by a known representation theorem [13, 26.2.8 and 26.2.10] we can identify B_e with $C^\infty(X_e)$. Here X_e is a basically disconnected compact Hausdorff space, $e \longleftrightarrow 1_{X_e}$, and $C^\infty(X_e)$ denotes the space of all extended real-valued continuous functions f on X_e such that $(t \in X_e : |f(t)| < \infty)$ is a dense cozero set in X_e. Let now $C(X_e)$ be the σ-Dedekind complete Riesz space of all real-valued continuous functions on X_e. Since the restriction of ρ to $C(X_e)$ is a σ-Lebesgue seminorm with $\rho(e) \neq 0$, by [3, 2.3] ρ has a compact support in X_e. Denote by C_ρ the compact support of ρ. It then follows that C_ρ is a finite subset of X_e. Indeed, if C_ρ is an infinite subset of X_e, then using the argument given in [5, 1.3] ([6, 2.4]) there exists a disjoint sequence $(U_n : n \in \mathrm{N})$ of clopen cozero sets of X_e such that $C_\rho \cap U_n \neq \phi$ for each n. Put

$$x_n = n 1_{U_n} / \rho(1_{U_n}) \text{ for each } n.$$

Since $1_{U_m} \wedge 1_{U_n} = \theta$ for $m \neq n$, $u = \sup_n x_n$ exists in $C^\infty(X_e)$, and so we get $\rho(x_n) = n \le \rho(u)$ for all n, which contradicts $\rho(u) < \infty$. Let $C_\rho = (t_k \in X_e\colon k = 1,2,...,n)$ and let V_k, $k = 1,2,...,n$ be disjoint clopen subsets of X_e with $t_k \in V_k$ for each k. Since $(x - (x | \cup_{k=1}^n V_k)) | C_\rho = 0$ for $x \in C(X_e)$, $\rho(x) = \rho(x | \cup_{k=1}^n V_k)$ holds. Here for $E \subset X_e$ and $x \in C(X_e)$, $x | E$ denotes a function on X_e that vanishes on the complement of E and is equal to x on E. Hence $\rho(x) \le \Sigma_{k=1}^n \rho(x | V_k)$ holds for all $x \in C(X_e)$. Further,

$$(x | V_k - x(t_k) 1_{V_k}) | C\rho = 0,$$

and so $\rho(x | V_k) = |x|(t_k) \rho(1_{V_k})$ holds for all $x \in C(X_e)$. For each k put $a_k = \rho(1_{V_k})$ and $\varphi_k(x) = x(t_k)$ for all $x \in C(X_e)$. Since ρ is σ-order continuous on $C^\infty(X_e)$, it follows easily that φ_k is a Riesz σ-homomorphism on $C(X_e)$. By Theorem 1 φ_k extends to a Riesz σ-homomorphism $\varphi_k^\sim$ on $C^\infty(X_e)$ and $\rho(x | V_k) = a_k \varphi_k^\sim(|x|)$ holds for all $x \in C^\infty(X_e)$. For every $x \in L^s$ put $\pi_k(x) = \varphi_k^\sim(P_e(x))$. Then it is clear that π_k is a Riesz σ-homomorphism on L^s. From this, it follows that $a_k \pi_k \leq \rho$ on L^{s+} holds for each k and $\rho \leq \Sigma_{k=1}^{n} a_k \pi_k$ on L^{s+}, and the restriction of the last inequalities to L^+ satisfy our requirement.

(3)$\Rightarrow$(4) Note first that by [1, 11.11] any σ-Lebesgue topology λ on L extends uniquely to a σ-Lebesgue topology λ^σ on L^σ. By way of contradiction, assume that $x_n \xrightarrow{\lambda} \theta$ in L, but $x_n \xrightarrow{\tau} \theta$ does not hold. Then we may consider that by passing to a subsequence (if necessary) there exist an $\varepsilon > 0$ and a $\rho \in S_\tau$ such that $\rho(x_n) > \varepsilon$ for all n. By (3) let π_k, $k=1,2,...,n$ be Riesz σ-homomorphisms on L such that $\rho \leq \Sigma_{k=1}^{n} a_k \pi_k$ on L^+ for some positive real numbers a_k, $k=1,2,...,n$. Choose an $x \in L^+$ such that $\pi_k(x) > 0$ for $k=1,2,...,n$. Let B_x the band of L^s generated by x, and by [13, 26.2.8 and 26.2.10] identify B_x with $C^\infty(X_x)$, where X_x is a basically disconnected compact Hausdorff space. By Theorem 1 each π_k extends uniquely to a Riesz σ-homomorphism $\tilde{\pi_k}$ on $C^\infty(X_x)$, and using the arguments given in [6, 2] it is not dificult to verify that there exists a P-point t_k of X_x such that $\tilde{\pi_k}(u) = u(t_k)$ for all $u \in C^\infty(X_x)$. Hence $\pi_k(x_n) = \tilde{\pi_k}(P_x(x_n)) = P_x(x_n)(t_k)$ holds for all $n \in \mathbf{N}$. Since $\varepsilon < \rho(x_m) \leq \Sigma_{k=1}^{n} a_k \pi_k(|x_m|)$ for all m, it follows easily that for some $\varepsilon' > 0$ there exist a π_k and a subsequence (x_{n_i}) of (x_n) such that

$$|\pi_k(x_{n_i})| = |P_x(x_{n_i})(t_k)| > \varepsilon' \text{ for all } i \in \mathbf{N}.$$

Put $G = \bigcap_{i=1}^{\infty}(t \in X_x\colon |P_x(x_{n_i})(t)| > \varepsilon')$. Since G is a G_δ-set of X_x containing t_k, by [8, 4L] there exists a clopen subset U of X_x with $t_k \in U \subset G$, and $\varepsilon' 1_U \leq P_x(x_{n_i})$ holds in L^σ. Since λ^σ is Hausdorff on L^σ, there exists a Riesz pseudonorm $\eta \in P_{\lambda^\sigma}$ such that $\eta(1_U) > 0$. But then $0 < \eta(1_U) < a\eta(x_{n_i}) \to 0$ holds for sufficiently large positive number a, which is a contradiction, and hence our result follows.

(4)$\Rightarrow$(5) Let $A \subset L$ be a λ-bounded subset of L. Then it is easy to show that for every sequence $(x_n) \subset A$ and every sequence $(\lambda_n\colon n \in \mathbf{N}) \subset \mathbf{R}$ with $\lambda_n \xrightarrow[n]{} 0$, $\lambda_n x_n \xrightarrow{\lambda} \theta$ holds. Now, if A is not τ-bounded, then for some $\rho \in S_\tau$ and $\varepsilon > 0$ there exists a sequence (y_n) of A such that $y_n/n \notin (x \in L\colon \rho(x) < \varepsilon)$ for each $n \in \mathbf{N}$. Since $y_n/n \xrightarrow{\lambda} \theta$ holds in L, it follows by (4) that $y_n/n \xrightarrow{\tau} \theta$, in contradiction to the choice of (y_n). Hence A is τ-bounded.

(5)$\Rightarrow$(6) Let A be a dominable subset of L. It is clear that A is also a dominable subset of L^σ. By [1, 23.12] there exists a complete disjoint system $(e_\alpha\colon \alpha \in \mathrm{I})$ of strictly positive elements of L^σ such that for each $\alpha \in \mathrm{I}$ there exists a positive integer n_α satisfying $P_\alpha(x) \leq n_\alpha e_\alpha$ for all $x \in A$. Here P_α is the projection from L^σ onto B_α and B_α denotes the band of L^σ generated by e_α. Let ΠB_α be the Cardesian product of $(B_\alpha\colon \alpha \in \mathrm{I})$. Then by [1, 3.2] L^σ can be embedded as an order dense Riesz subspace in ΠB_α. Let τ^σ be a unique σ-Lebesgue topology on L^σ that is an extension of τ. For each α let τ_α be the Hausdorff σ-Lebesgue topology of the restriction of τ^σ to B_α. Then it follows by [1, 5.5] that $(\Pi B_\alpha, \Pi \tau_\alpha)$ is a Hausdorff σ-Lebesgue Riesz space, and its restriction to L is also a Hausdorff σ-Lebesgue topology. It is simple to verify that A as embedded in ΠB_α is $\Pi \tau_\alpha$-bounded. Hence it follows by our hypothesis that A is τ-bounded.

(6)$\Rightarrow$(1) Let $\theta \leq u \in L^s$. Using the argument given in [1, 23.13] it is clear that L_u is a dominable subset of L, and hence by (6) L_u is τ-bounded. Let $\rho \in S_\tau$ and for each $u \in L^s$ define $\tilde{\rho}(u) = \sup_n \rho(x_n)$ by a sequence $(x_n) \subset L_{|u|}$ with $\theta \leq x_n \uparrow |u|$ in L^s. It is simple to verify that $\tilde{\rho}$ is well-defined and a Riesz seminorm on L^s. Let τ^s denote a locally convex-solid topology on L^s generated by the family $(\tilde{\rho}\colon \rho \in S_\tau)$. Then τ^s is a Hausdorff topology on L^s that induces τ on L, and the proof is finished.

Similarly, the next result tells us what Hausdorff locally solid σ-Lebesgue topologies on L extend to a locally solid topology on L^s, and it suggests a sequential version of theorem 3.7 given in [2] ([1, 24.3]).

THEOREM 3. *Let L be an almost σ-Dedekind complete Riesz space and let τ be a Hausdorff locally solid σ-Lebesgue topology on L. Then the following statements are equivalent*:

(1) *τ extends to a locally solid topology on L^s.*

(2) *τ extends to a locally solid topology on L^s having both the σ-Lebesgue property and the σ-Levi property.*

(3) *For any Hausdorff σ-Fatou topology λ defined on L, every λ-bounded sequence (x_n) of L with $\theta \leq x_n \uparrow$ in L implies τ-bounded.*

(4) *Every countable dominable subset of L is τ-bounded.*

Proof. (1)$\Rightarrow$(2) By [1, 24.1] and [4, 1.4] it is obvious.

(2)$\Rightarrow$(3) Let λ be a Hausdorff σ-Fatou topology on L and assume that $\theta \leq x_n \uparrow$ in L is λ-bounded. Then by [9, Th. A] there exists the largest σ-enlargement $(\tilde{L}, \lambda^{\sim})$ of (L, λ) such that $(\tilde{L}, \lambda^{\sim})$ is a σ-Nakano space, where $L^{\sigma} \subset \tilde{L} \subset L^s$. Hence $x = \sup_n x_n$ exists in $\tilde{L}$ and so it follows immediately that (x_n) is τ-bounded.

(3)$\Rightarrow$(4) By [1, 11.11] let τ^{σ} be a unique σ-Lebesgue topology on L^{σ} that induces τ on L. Let (x_n) be a countable dominable subset of L^+. Then by [1, 23.10] (x_n) is also a dominable subset of L^s, and hence by [1, 23.23] (x_n) is order bounded in L^s. Without loss of generality, we may assume that (x_n) is increasing in L. By [1, 23.12] there exists a complete disjoint system $(e_{\alpha}\colon \alpha \in \mathrm{I})$ of strictly positive elements of L^{σ} such that for each $\alpha \in \mathrm{I}$ there exists a positive integer n_{α} satisfying $P_{\alpha}(x_n) \leq n_{\alpha} e_{\alpha}$ for all n. Here P_{α} is the projection from L^{σ} onto B_{α} and B_{α} denotes the band of L^{σ} generated by e_{α}. By [1, 3.2] embedd L^{σ} as an order dense Riesz subspace in ΠB_{α}. Let τ_{α} be the Hausdorff σ-Lebesgue topology of the restriction of τ^{σ} to B_{α}. Then it follows by [1, 8.12] that $\Pi \tau_{\alpha}$ is a Hausdorff σ-Lebesgue topology on ΠB_{α}, and hence its restriction to L defines also a Hausdorff σ-Lebesgue topology λ. It is easy to show that (x_n) as embedded in ΠB_{α} is $\Pi \tau_{\alpha}$-bounded. Hence (x_n) is λ-bounded and the result now follows immediately from (3).

(4)$\Rightarrow$(1) Let $\theta \leq u \in L^s$. Choose a sequence (x_n) of L^+ with $\theta \leq x_n \uparrow u$ in L^s. Then using the argument given in [1, 23.13] it is clear that (x_n) is a dominable subset of L. Hence by (4) (x_n) is τ-bounded. Let $\rho \in \mathrm{P}_{\tau}$ and for each $u \in L^s$ put $\tilde{\rho}(u) = \sup_n \rho(x_n)$, where $(x_n) \subset L_{|u|}$ and $\theta \leq x_n \uparrow |u|$ in L^s. It is simple to verify that $\tilde{\rho}$ is well-defined and a Riesz pseudonorm on L^s. Let τ^s be the topology on L^s generated by the family $(\tilde{\rho}\colon \rho \in P_{\tau})$. Then τ^s is a unique Hausdorff locally solid topology on L^s that induces τ on L, and the proof is finished.

As corollary of Theorem 3, the following result answers the question when a Hausdorff locally solid σ-Lebesgue topology τ on a σ-Dedekind complete Riesz space L extends to a locally solid topology on L^s.

COROLLARY 4. *Let L be a σ-Dedekind complete Riesz space and let τ be a Hausdorff locally solid σ-Lebesgue topology on L. Then following are equivalent*:

(1) *τ extends to a locally solid topology on L^s.*

(2) *τ extends to a locally solid topology on L^s having both the σ-Lebesgue property and the σ-Levi property.*

(3) *For any Hausdorff σ-Fatou topology λ defined on L, every λ-bounded sequence (x_n) of L with $\theta \leq x_n \uparrow$ in L implies τ-bounded.*

(4) *Every countable dominable set of L^+ is τ-bounded.*

(5) *For each disjoint sequence (x_n) of L, the set $(\Sigma_{i \in F} x_i : F \subset \mathbf{N}$ and F is finite) is τ-bounded.*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) Obvious from Theorem 3.

(4)$\Rightarrow$(5) Let (x_n) be a disjoint sequence of L^+ and put A$=(\Sigma_{i\in F} x_i : F\subset \mathbf{N}$ and F is finite). Since $u = \sup_n x_n = \sup_n \Sigma_{k=1}^n x_k$ exists in L^s, it follows by the argument given in [1, 23.13] that A is a dominable subset of L. Hence by our hypothesis A is τ-bounded.

(5)$\Rightarrow$(1) Note first that by [9, Th. A] there exists the largest σ-enlargement $(\tilde{L}, \tau^{\sim})$ of (L, τ) such that $(\tilde{L}, \tau^{\sim})$ is a σ-Nakano space, where $L^{\sigma}\subset \tilde{L}\subset L^s$. Let $\theta \leq u \in L^s$ and choose a sequence (x_n) of L^+ with $\theta \leq x_n \uparrow u$ in L^s. Let $y_1=x_1$ and then inductively if $y_1, y_2, \cdots, y_n$ have been defined in L, put $z_n=y_1+y_2+\cdots+y_n$ and define $y_{n+1}=x_{n+1}-P_n(x_{n+1})$ in L, where P_n is the projection from L^s onto B_{z_n} and B_{z_n} denotes the band of L^s generated by z_n. Then it is easy to show that $(y_n: n\in \mathbf{N})$ is a disjoint sequence of L. Hence by (5) a sequence (z_n) is τ-bounded and so $e=\sup_n z_n(=\Sigma_{n=1}^{\infty} y_n)$ exists in $\tilde{L}$. Let B_e be the band of L^s generated by e. It then follows immediately that $(x_n)\subset B_e$, and hence $u\in B_e$. Now, by [13, 26.2.8 and 26.2.10] we identify B_e with $C^{\infty}(X_e)$. Here X_e is a basically disconnected compact Hausdorff space and $e \longleftrightarrow 1_{X_e}$. For each n put

$$U_n = \mathrm{cl}(t\in X_e : 2x_n(t) > u(t)), \text{ where "cl" is the closure in } X_e.$$

Obviously, each U_n is a clopen subset of X_e. Put $E_1=U_1$ and $E_{n+1}=U_{n+1}\setminus U_n$ for each n. Then it is not difficult to verify that a sequence $(u1_{E_n})$ is a disjoint sequence of $B_e\cap L^+$. Hence by (5) $(\Sigma_{i\in F} u1_{E_i}: F\subset \mathbf{N}$ and F is finite) is τ-bounded, and so $u=\sup_n u1_{E_n}(=\Sigma_{n=1}^{\infty} u1_{E_n})$ exists in $\tilde{L}$. This means that $L^s=\tilde{L}$, and the proof of the corollary is now complete.

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